COMPLICATED ONE-DIMENSIONAL FLOWS

by

Gino Moretti

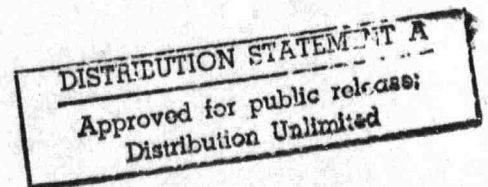# POLYTECHNIC INSTITUTE OF BROOKLYN

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13. ABSTRACT

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ABSTRACT

A computational technique for one-dimensional, unsteady flow is
presented. The technique emphasizes the role of discontinuities (shocks,
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practical arguments to support such an approach are given. The technique
allows the computational time to be reduced to a minimum. The accuracy
of the results as well as the generality of the program are shown by many
applications to one-dimensional and quasi-one-dimensional problems.

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One-dimensional flows, in the present paper, are unsteady flows
which depend on one space variable only. Therefore, unsteady flows
whose properties are uniform along planes, cylinders or spheres, as well
as flow in ducts of variable cross-section, treated within the framework
of the well-known quasi-one-dimensional approximation, all belong to that
category. Despite their crudity, one-dimensional concepts find useful
applications in a large number of problems, such as shock-tube analysis,
intake and exhaust unsteady behavior, preliminary wind-tunnel design,
explosions, stellar evolution, vehicle-in-tube performances, etc. In
addition, one-dimensional concepts can be used, by analogy, in steady,
supersonic flows depending on two space variables (that is, two-dimen-
sional or axially symmetric). From the viewpoint of research, in gas
dynamics as well as in numerical analysis, one-dimensional problems pro-
vide basic examples in the simplest possible way, clear-cut ideas and
accuracy tests.

Has a problem with so many positive features and so rich in practical
applications been formulated in a general, exhaustive way? Is a computer
program available, easy to use, general, safe, accurate and fast?
Judging by the existing literature and the requests from the industry,
the answer seems to be on the negative.

The reason is that one-dimensional problems are not simple at all.
As a consequence of the nonlinearity of the governing equations, discon-
tinuities in the physical parameters and their derivatives appear, most
disturbingly in the form of shock waves and interfaces (contact discon-
tinuities). Such discontinuities interact with each other, generating

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more discontinuities, and reflect on boundaries. In general, flows which start with a continuous distribution of parameters, quickly develop discontinuities. A cross-section of the flow at a constant time appears as a complicated distribution of discontinuities, separating diminutive regions of continuous flow.

To make the point bluntly: Rather than as a continuous distribution of parameters, occasionally interrupted by discontinuities, <u>a one-dimensional flow can be conceived as a pattern of discontinuities, separating regions of almost uniform flow</u>. Bearing this in mind, a computer program should emphasize the handling of discontinuities and their interactions, rather than the calculation of continuous regions.

The trend in the last two decades has been in the opposite direction. It is, by now, so deeply rooted that few words to explain how the shift began may hopefully help to redirect our efforts onto what I believe is the right track. The first attempts to calculate one-dimensional flows were based on the method of characteristics, a convenient technique for hand computations and graphical analysis as long as the flows are continuous, homoentropic and strictly one-dimensional. In the presence of discontinuities, the method of characteristics has to be modified. The latter can be used in each region of continuous flow, with the additional complication of a generally variable entropy; the discontinuities must be treated by different techniques. Hand computations become prohibitively cumbersome; interpreting the method in a computer program is a formidable exercise in logic. Not surprisingly, scientists interested in fluid mechanics shied away from such problems, whose solution seldom contributes to deepening one's physical knowledge. On the other hand, skilled

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programmers who may enjoy a challenge in logic cannot undertake the task without a gas dynamicist's guidance. In 1950, von Neumann and Richtmyer came along with a brilliant idea[1], which seemed to sweep out all logical difficulties by a single stroke, making gas dynamical computations an effortless routine, to be stolidly performed by the high-speed computer, over and over again, invariably, at all the points to be evaluated. In the scientific community, such a technique soon became known as "brute force", the derogatory tinge being intentional. The industry, however, welcomed it as a simple way of obtaining results, with a minimal contribution of specialized manpower, and of replacing expensive and delicate experiments by inexpensive and safe computations. The optimistic term, "electronic wind tunnel", was coined for the high-speed computer working on a gas dynamical problem.

It turned out that the computations were <u>neither safe nor inexpensive</u>. One-dimensional problems clearly show why the method fails. To eliminate discontinuities, one has to smear them out by introducing an artificial viscosity into the finite-difference equations. If the number of mesh points is too small, the artificial viscosity is too high. Thus, discontinuities diffuse over too wide a region. When more than one discontinuity exist, their broadened counterparts tend to overlap and the intermediate region is completely defaced. A possible remedy consists of increasing the number of mesh points. The points must be evenly distributed all over the region to be computed since (according to the "brute force" approach) one does not know where the discontinuities are going to be at any given time; and the distance between mesh points has to be made <u>really</u> small. Computations which may claim to be safe (that is, accurate)

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are catastrophically expensive[2,3].

FIG. 1. COMPARISON BETWEEN SOLUTIONS OBTAINED BY

CHARACTERISTIC AND MESH CODES (FROM REF. 4)

A couple of examples will help make the point clear:

1) See Fig. 6 of Ref. 4 (reproduced here as Fig. 1), where the
dotted line plots the results of a brute force method. To quote the
authors of Ref. 4: "This calculation used 470 points and took three times
as long as the characteristic calculation (15 minutes instead of 5[*]). It
will be seen that there is a larger uncertainty in the shock positions
and strengths. Smoother profiles can be obtained by using different
artificial viscosities, but at the expense of further shock broadening and

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[*]The computer used in the test is not mentioned.

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a corresponding increase in uncertainty." I agree; I would like to anticipate, though, that by using the method explained in the present paper a maximum of 35 points could give accurate results, without certain complications inherent in the method of characteristics. The reduction in computational time would thus be by a factor of one hundred or more.

2) Ref. 5, instead, is a typical application of brute force methods with no imaginative criticism. After stating that "the availability of larger, faster computing machines now allows the complete governing non-linear equations to be solved by conceptually simple and more generally applicable explicit finite difference methods," the author proceeds to apply the obsolete Lax scheme, notorious for its low degree of accuracy. The ratio of (CDC 6600) computer time to real time is, according to Ref. 5, about 6000 for relatively simple problems (constant area duct, a single shock), and should be multiplied by 4 for more complicated problems. By the method of the present paper, the example which requires 200 mesh points in Ref. 5 could be analyzed with about 20 points, thus reducing the computational time by a factor of 100. The ratio of computer time to real time would then be only 60, and the accuracy would be improved as well. Note that such a ratio is still unacceptable for many practical applications. However, a machine ten times as faster as the CDC 6600 would be sufficient to make the computational time acceptable, and this is within our immediate reach. It should also be noted that, for problems complicated by the presence of many discontinuities, the total number of points remains substantially unchanged, if the method of the present paper is used, whereas it is multiplied by a sizeable factor (to be squared to evaluate the increase in computational time) if the method of Ref. 5 is used.

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Along the same line of thought, we find in the more recent litera-
ture methods which seem capable of a greater accuracy than Lax's scheme,
claiming not to use artificial viscosity but, nevertheless, to be able to
find and handle discontinuities without explicit provisions. One of such
methods has been authoritatively supported in connection with other prob-
lems of a similar nature[6]. The technique has been labeled "shock
capturing" for its purported ability to build up sharp transitions,
roughly equivalent to shocks, on about three mesh points. In problems as
complicated as the ones discussed in the present paper where, as we will
see, regions described by only three points between discontinuities exist,
one may wonder how the two limiting discontinuities would appear.

We are back again to a need for more points! However, regardless of
the number of points, the method seems to be inconsistent because, if
viscosity is completely eliminated, the equations of motion are not prop-
erly used in the transition substituting for the discontinuity (see Ref. 2,
pages 19 and 54). In the same Ref. 2, page 50, I have mentioned McCormack's
scheme as a fairly good second order method, and elsewhere I have repeatedly
pointed out some of its practical coding advantages in complicated, multi-
dimensional problems. I cannot share, though, the optimism of the authors
of Ref. 6 in saying that McCormack's scheme is a "shock capturing" one,
more than other schemes and within acceptable time limitations.

To submit my theoretical arguments of Ref. 2, page 54 and Lomax and
Kutler's opposite statements to a practical test, we may resume the cal-
culation which in Ref. 2 was halted soon before the appearance of an
imbedded shock. According to the theoretical analysis of Ref. 7, page 38,
the shock should form at t=.844, x=.8415. Results obtained by using

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McCormack's scheme, with the original equations either in conservation form or not, and Δx equal to .05, .025, .0125, .00625 show no differences from each other as long as t does not exceed .8.

If t increases beyond .8, the results obtained when McCormack's scheme is applied to the non-conservative form of the equations oscillate in the region where the shock should appear. The oscillating pattern worsens by refining the mesh, a feature which makes any attempt to better accuracy hopeless.

Better results, on the high pressure side of the shock, are obtained if the equations are recast in conservation form. The low pressure side, however, does not improve. Fig. 2 shows the velocity distribution at t=.901, as computed by the McCormack scheme with the equations in conservation form, with Δx=.001667 (360 points), and with Δx=.013333 (45 points), together with the velocity distribution at the same time, computed by the method of the present paper, using a total of 20 points (Δx equals .03 before the formation of the shock and it maintains a value of the same order of magnitude after the shock is formed). The shock location, properly found by the present technique with 20 points, is detected by the "shock capturing" scheme if 360 points are used. The low pressure side distribution is misrepresented even with 360 points (note that in this problem the velocity in front of the shock is small but positive, as the exact solution and our results show). One can anticipate some embarrassment, should another shock, proceeding from right to left, impinge on the former.

Anyway, the ratio of computational time between the 360 point case and the 20 point case is 329 ($\frac{360}{20}$ x $\frac{1700}{93}$, the second fraction being the
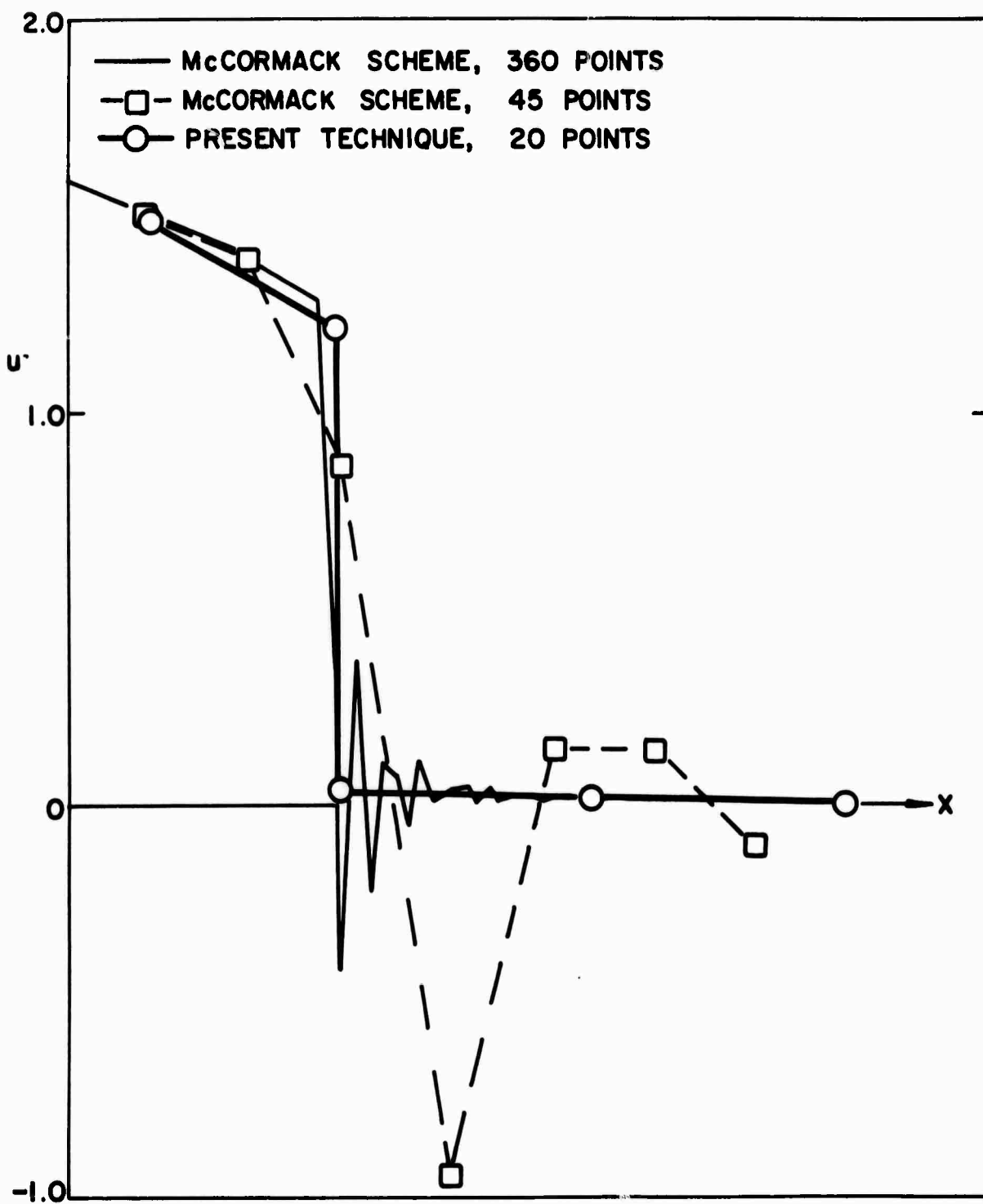
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FIG. 2. EXPLICIT SHOCK VS. SHOCK-CAPTURING RESULTS

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ratio of time steps to reach t=.901). With such a ratio, assuming that the present technique can handle a problem at a ratio of computational time to real time equal to 40, the same ratio for the shock-capturing technique becomes 13160, twice as larger as the one mentioned in Ref. 5.

II. A MARCH-ON TECHNIQUE EMPHASIZING THE ROLE OF DISCONTINUITIES

The present paper introduces a computational technique which takes full advantage of explicit treatment of discontinuities. Although the current version of the code uses the concept of characteristics occasionally, its general logic is not based on the method of characteristics, as in Ref. 4. The belief that only if the method of characteristics is used, discontinuities can be treated explicitly is unjustified and only attributable to the historical, parallel development of the method of characteristics and finite-difference schemes as outlined above. The physical problem is sufficiently complicated per se, not to ask for the unnecessary logical complications brought in by the method of characteristics. Moreover, it must be noted that the latter does not offer any advantage as far as accuracy and computational speed are concerned.

The additional complications mentioned above proceed from the intrinsic irregularity of a characteristic network. In addition to making plotting very cumbersome, the characteristic pattern can be so different in different regions of the (x,t) plane that, in the course of computing, uninteresting regions or regions where the computation may fail for lack of accuracy are reached when the region of interest has not been fully explored yet. Finally, if the computation proceeds building up a right-running characteristic, say, a left-running shock can be evaluated without

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excessive additional labor; a right-running shock, however, cannot be evaluated without first determining a sufficiently wide, shockless region across which the shock has to be laid, thus canceling the portion of the computed region on its high pressure side. Logical difficulties arise and grow bigger and bigger as the discontinuities interact with each other. In addition, time is lost in computing parts which are successively erased. Finally, since the latter have no physical meaning, out-of-range numbers can be produced and the computation halted.

None of the above difficulties exists if one proceeds from a given time to a successive time, advancing by the same amount Δt at all points. This is easily and accurately done by using a suitable finite-difference scheme at all points where the physical parameters are continuous and differentiable, and their first derivatives are continuous.

However, according to the spirit of the present technique, I will postpone the discussion of the finite-difference scheme and discuss first the physical features of discontinuities and their interaction.

III. SHOCKS, CONTACT DISCONTINUITIES AND GRADIENT DISCONTINUITIES

Consider the flow described by three physical parameters, the particle velocity, u, the logarithm of pressure, P, and the entropy, S.

A shock is a locus of discontinuities for u, P and S. It travels at a speed, W, which, relatively to the moving particles, is always greater than the speed of sound on the low pressure side. Seven equations are needed to determine the values of u, P and S, on either side of the shock, plus W.

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A contact discontinuity is a locus of discontinuities for S alone. It travels with the particles, that is, at a speed equal to u. Therefore, four equations are needed to determine u, P, and the two values of S at either side of the discontinuity.

A gradient discontinuity is a locus of discontinuities for the first derivatives of u, P and S. It travels as a characteristic. At a point located on a gradient discontinuity there are no more unknowns than at an ordinary point. However, one cannot replace a derivative by a finite difference involving points at both sides of the line.

If each discontinuity is considered as a boundary point between two regions of continuous flow, we may count such a point twice, once as being the last point in the region at the left and again as being the first point in the region at the right.

Suppose the discontinuity is a shock. Since the flow in the low pressure region, relative to the shock, is supersonic, the flow values at time t are necessary and sufficient to determine the shock point on the same side at t+Δt, provided that its location is known. Both (u+a) and (u-a) characteristics, indeed, reach the shock from the low pressure side, and so does the particle path. Any three independent equations (such as the two compatibility equations along characteristics and the equation expressing the constancy of S along a particle path, or Euler's equations in their original form) provide the values of P, u, and S at the shock point on the low pressure side. From the high pressure side only one characteristic reaches the shock. The compatibility equation written for that characteristic, and the three Rankine-Hugoniot conditions across the shock are the necessary and sufficient equations to determine

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the values of P, u, and S at the shock point on the high pressure side, plus the shock velocity, W.

This basic idea can be applied in many different ways, with different degrees of accuracy and sophistication. Not all schemes fit equally well into the general logic of a program. The present code (December 1970), which was written with a well-separated treatment of every single item for a clearer understanding, would not easily accept the procedure suggested in Ref. 8 (the seven equations, plus an equation defining the shock path, are solved simultaneously by a two-level scheme which provides an overall second order accuracy). The procedure of the present code is conceptually simpler; however, the accuracy is only of the first order and some iterations are required.

The computation proceeds as follows. First, the location of the shock, x_s, is obtained at time $t+\Delta t$ by integrating the equation:

$$\frac{ds_x}{dt} = W$$

with the first-order approximation:

$$x_s(t+\Delta t)=x_s(t)+W(t)\Delta t$$

Then the low pressure region is identified; the two characteristic and the particle path are traced back from $x_s(t+\Delta t)$ into the low-pressure region. Their locations at time t are found and values of P, u, and S are obtained by linear interpolations on the values at mesh points at time t. The compatibility equations are applied to find the shock values at time $t+\Delta t$ on the low pressure side. Finally, the Rankine-Hugoniot conditions and the compatibility equation along the characteristic on the

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high pressure side are solved by a trial-and-error procedure.

By so doing, the shock computation depends only on values at interior points known at time t. It is thus explicit, that is, uncoupled from any other computation and it can be performed in an independent subroutine.

In the same spirit, the contact discontinuity can be evaluated as follows. First, the location x_c of the discontinuity at time $t+\Delta t$ is determined by integrating the particle path equation:

$$\frac{dx_c}{dt} = u$$

with the first-order approximation:

$$x_c(t+\Delta t) = x_c(t) + u(t)\Delta t$$

Two characteristics reach point x_c from either side of it. Their two compatibility equations are solved simultaneously, considering that P and u are the same on both sides of the discontinuity. The values of S on either side are carried over from t to $t+\Delta t$, unchanged.

Once more, it can be observed that a more accurate, two-level scheme could be used; but, again, the present procedure makes the computation explicit and it is better suited for the code in its present form.

Finally, for the gradient discontinuities two cases must be distinguished. If the discontinuity propagates an expansion, the path of a gradient discontinuity is defined by

$$\frac{dx_q}{dt} = u \pm a$$

This equation, once the proper choice of the sign is made, can be

integrated by a first order approximation as the others for x_s and x_c above. The values of P, u, and S at both sides of the discontinuity must be computed by using information only from the side from which the particles arrive. In fact, of the two characteristics, one is the discontinuity itself and the other carries information to x_g from one side and carries information away from x_g on the other side; similarly, the entropy maintains at x_g the value which it has before the particle reaches x_g. On these grounds, it is easy to work out a routine for an explicit treatment of x_g.

If the discontinuity propagates a compression, a shock will eventually form. Therefore, it is advisable to treat the discontinuity as a shock of zero strength.

IV. EXTINCTION OF DISCONTINUITIES

In an inviscid flow, discontinuities seldom disappear spontaneously. A shock may be extinguished, for example, by letting it reach a wall capable of absorbing its full impact at the right time, a condition hard to satisfy in a numerical computation. A contact discontinuity can never be eliminated, in principle.

Although the program is capable of handling a great number of discontinuities, there is no reason for carrying them along and letting them multiply ad infinitum, unless their role is significant. Provisions are taken in the program to eliminate shock waves whose pressure ratio is below a certain tolerance, contact discontinuities whose temperature ratio is below a certain tolerance, and gradient discontinuities where the difference between the values of $\partial P/\partial x$ on either side is below a certain

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tolerance. Tolerances can be prescribed by the user according to his needs.

In any case, once a discontinuity is eliminated, the two points from either side of it are merged into a single point and all mesh points are redistributed uniformly across the new region. Physical values at the new mesh points are linearly interpolated from the data available immediately prior to the merging.

V. INTERACTION OF DISCONTINUITIES

It is well-known that the way discontinuities interact with each other depends on local properties of the flow in the vicinity of the point where such discontinuities meet. Consequently, a general analysis of interactions can be made and has actually been made in the forties. Its results are now available in textbooks at the graduate level and I do not consider necessary to repeat them here. A computational code may take care of the interactions by:

1. Determining that an interaction should occur within the step to be taken, and

2. Performing a local study of the flow field, in which the minute regions between discontinuities are assumed to be regions of uniform flow.

The first problem is easily solved since the present code keeps track of the size of the regions between discontinuities and of the slopes of the discontinuities.
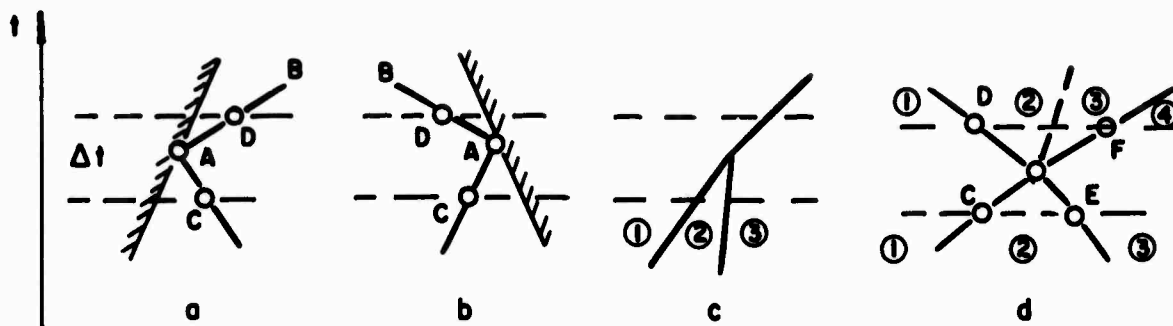
FIG. 3.   SHOCK REFLECTIONS AND INTERACTIONS

Figure 3 shows the interactions involving impinging shocks only, considered by the code. Cases a and b are reflections of shocks on rigid, moving walls. The values at D are assumed equal to the values at C. The slope AB of the reflected shock is, by trial and error, obtained to satisfy the Rankine-Hugoniot conditions across the shock at D, under the assumption that behind the shock the velocity of the particles equals the velocity of the wall. Note that, after reflection, the number of regions of continuous flow is still the same as before reflection.

Case c represents the coalescence of two shocks of the same nature. In this case the code simply assumes that the two shocks merge. Region 2 between them is eliminated, and the computation proceeds by determining the location and slope of a shock separating region 1 from region 3, as outlined in Section III. Any expansion wave produced in the interaction is sufficiently weak to be computed by the code for continuous regions without additional sophistication. The contact discontinuity which should

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emerge from the merger point is considered negligible.

In case d, two shocks of opposite nature impinge upon each other, are refracted and a contact discontinuity is generated. Again, the values at D and F are assumed equal to the values at C and E, respectively. The slopes of the refracted shocks and of the contact discontinuity, as well as the (uniform) values of the flow field parameters in regions 2 and 3 behind the merger point are obtained by trial and error; the Rankine-Hugoniot conditions across both refracted shocks and the conditions of equal P and u in regions 2 and 3 are used. Note that in this case, after reflection, the number of regions of continuous flow is increased by one.

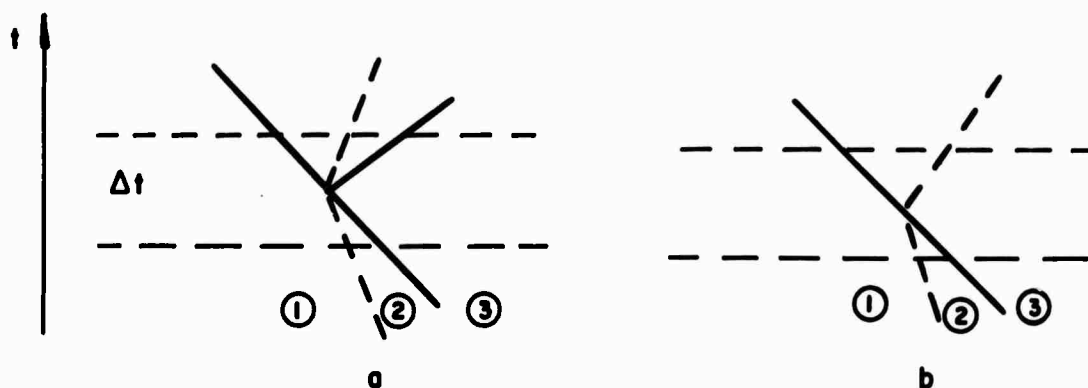
FIG. 4. SHOCK-CONTACT DISCONTINUITY INTERACTIONS

A contact discontinuity cannot reach a rigid wall. It can interact with a shock in two ways, represented in Fig. 4. The different pattern is a consequence of the ratio of densities across the contact discontinuity before it crosses the shock. Pattern (a) (refracted shock, refracted contact discontinuity and a reflected shock) occurs if the

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density in region 1 is greater than the density in region 2. In the opposite instance, pattern (b) occurs (no reflected shock). The handling of such cases follows the outline for shock interaction very closely.

VI. GENERATION OF DISCONTINUITIES

In the present code, which does not consider layers of different gases, contact discontinuities can only be generated by shock interaction, as outlined in Section IV. Shocks are generated by abrupt changes in the speed of a piston, or by coalescence of characteristics. In the majority of problems, the time and place where coalescence occurs are not directly related to typical geometrical features of the duct or, if moving pistons exist, to their accelerations. A method to find the birthplace of an imbedded shock by analyzing the shape of the pressure distribution has been outlined in Ref. 7; it is applied in the present code and, in all numerical experiments performed so far, has proved to be efficient.

If the piston velocity changes abruptly at time t, producing a sudden compression, a shock originates at the piston itself at time t. In this case, a new region of continuous flow behind the shock must be generated. The procedure is easily established by introducing a fictitious shock of zero strength impinging on the piston at time t and considering the new shock as a reflection of the former one (case a or b of Fig. 3).

Finally, the piston acceleration may change abruptly at time t, producing a compression. In this case, the flow velocity remains continuous until a shock forms by coalescence of characteristics. The derivatives

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of velocity and pressure, though, are discontinuous along the characteristic issuing from the piston at time t; in other words, the shock will eventually form along a gradient discontinuity. One may use the technique mentioned above to find imbedded shocks in this case. However, unless the characteristic itself is considered as a boundary between two regions, the calculation may become affected by gross inaccuracies long before the shock has a chance to be detected. It is much better, thus, to locate the point of discontinuous acceleration in the piston and to consider a shock issuing from that point, by proceeding as explained above for the case of discontinuous piston velocity. Since the velocity is not discontinuous, the shock will automatically begin with zero strength and proceed for awhile along a characteristic. The advantage in so doing is that the characteristic will be singled out from its inception as a boundary between two continuous regions, and the computation along the characteristic will have all the features of a shock computation. When coalescence occurs, the yet infinitely weak shock will be able to pick up strength and the neighboring points will always be computed without carrying finite differences across discontinuities of any type. An example of the improvement in accuracy obtained by fitting a shock at a point of discontinuous acceleration will be given in Section IX.

It should be noted that proper identification of the starting point of a shock is imperative to obtain accurate results in the adjoining regions of continuous flow. If the shock is fitted too far from the place where it would actually originate, it will not fulfill its mission; another shock will tend to form in the right place, with the consequent loss of accuracy exposed in Section I. An example of this kind of

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inaccuracy is given in Ref. 9, whose Figs. 27 and 28 are reproduced here in Fig. 5. Figure 28 shows (in heavy lines) a piston path and an assumed shock path, whereas the actual shock should be the light line issuing from x=3.0948, t=1.5. Figure 27 shows successive pressure distributions in the region between the piston and the assumed shock. Eventually, (at t=2.23) the real shock overtakes the assumed shock and the accuracy improves. However, the transient is poorly described (note all the disturbing oscillations which a well-fitted shock would eliminate). Note also that the number of mesh points (not mentioned in Ref. 9) seems to be quite large, whereas the method of the present paper handles the same problem with only three points between piston and shock.

If the shock were fitted too close to the origin of compression (the piston), the shock could not function properly as a relieving mechanism for the pressure surge, and again, the pressure distribution would become accordion-pleated until the shock had a chance to move into its right location. All these inaccuracies become very important and, quite often, catastrophic in problems involving multiple shocks and their interactions.

VII. TREATMENT OF CONTINUOUS REGIONS

Having thus described briefly how the most relevant features of one-dimensional flows can be treated, it remains to say how the computation may be performed in regions of continuous flow. I have pointed out in Ref. 2 that a second-order accurate scheme for integrating Euler's equations is a reasonably good choice, and I have expressed my belief that the most popular schemes are almost equivalent, as far as accuracy is
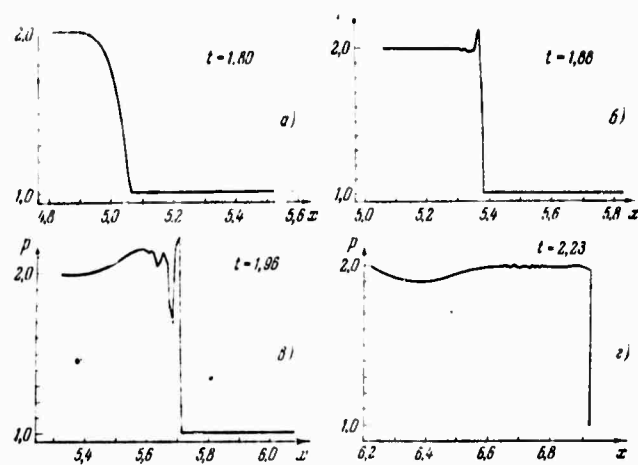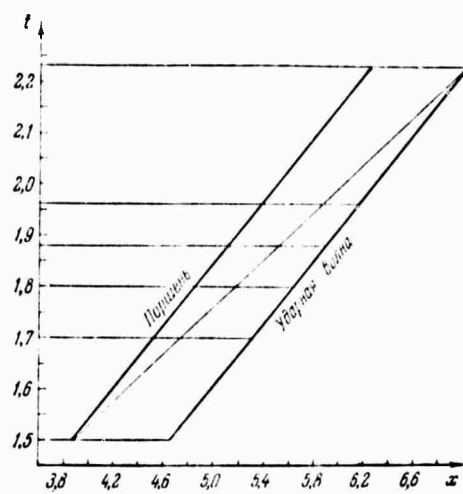
Рис. 27.

Рис. 28.

FIG. 5. COMPUTATION OF THE FLOW PRODUCED BY A PISTON

SUDDENLY SET INTO MOTION, ACCORDING TO REF. 9

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concerned. The present code, written in its original form a couple of years ago, uses the scheme No. 6 of Ref. 2, page 42. Such a scheme has the advantage of being symmetrical; in a problem where two pistons move symmetrically at both ends of a duct, the results inside are symmetrical as well. MacCormack's scheme (No. 7 of Ref. 2, page 42) would not provide such a symmetry, except for few of the first significant figures. The latter, however, has some advantages in coding simplicity and, when properly combined with schemes to treat discontinuities, can provide some reduction in computational time. Therefore, a new version of the code will be provided eventually, with a faster working logic, exploiting the positive features of MacCormack's scheme.

The equations of motion are written for the case of a duct of variable cross-section in the form:

$$P_t + uP_x + \gamma u_x + \gamma u\alpha = 0$$

$$u_t + uu_x + \Im P_x = 0$$

$$S_t + uS_x = 0$$

where P is the logarithm of pressure, u the velocity, $\Im$ is the ratio of pressure to density and S is the entropy. All these quantities are made nondimensional as explained in Ref. 2, page 22 or in a similar manner (for inlet problems) by assuming p_{ref} and ρ_{ref} as the values of pressure and density at the entrance section of the duct. The quantity α is the logarithmic derivative of the cross-sectional area with respect to x. In each region of continuous flow, the equations are normalized by assuming new variables x and T,

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$$X = \frac{x-b}{c-b}$$

$$T = t$$

where b and c are the left and right boundary of the region, respectively.
The finite-difference scheme is then applied to the equations in the
(X,T) space.

VIII. INITIAL CONDITIONS, BOUNDARY CONDITIONS
AND DISTRIBUTION OF MESH POINTS

Everyone who has ever written a computational code knows how diffi-
cult it is to make it flexible and general. Regardless of the care taken
in the planning stage to give the code some generality, soon a request
for application will show features of the geometry, boundary conditions
and/or initial conditions which had been overlooked. One of the advan-
tages of the present approach proceeds from its segmentary structure.
The basic routines (shock detection, shock fitting, contact discontinuity
fitting, gradient discontinuity fitting, reflections and interaction of
discontinuities, treatment of continuous regions) are independent from
one another and can be used with no changes, regardless of the number of
discontinuities and of the nature of initial and boundary conditions.
Nevertheless, I disclaim any attempt to generality in the present stage
of evolution of the code. Originated two years ago as a computational
tool for designing a shock-tube device known as the Slingshot[10], the
code was limited to a constant area duct, bounded by two moving pistons;
the code included provisions for detection of imbedded shock, shock

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fitting, reflection and interaction (without contact discontinuities).
In its present form, the code can handle all the features described in
the preceding sections. As far as the boundaries are concerned, any
combination of the following conditions can be handled:

a) a moving piston,

b) an arbitrary boundary (either fixed or moving with a prescribed
law) in a region of invariable flow,

c) an infinite capacity containing gas at rest,

d) a prescribed supersonic flow,

e) an infinite cavity in which the gas issues as a jet,

f) an interface with an engine, whose characteristics are to be
prescribed by an additional program.

The initial conditions, which were limited to a gas at rest in the
original version of the code, may now be:

a) gas at rest,

b) steady flow in a duct of variable cross-section, with or without
a normal shock, obtained automatically by prescribing the exit pressure.

Once the boundary and initial conditions have been chosen by reading
in three code integers, and supplementing the duct geometry and the law
of motion of the external boundaries, it remains to choose the number of
mesh points in the initial configurations. The choice is dictated by the
duct geometry and the initial distribution of parameters. If the gas is
initially at rest, the duct has a constant area and the limiting pistons
move to produce compressions, the code will immediately split the initial
region into three parts, separated by shocks, so that the initial region
will still contain gas at rest until the two shocks interact with each

other. In this case, the number of initial points may be limited to three. If the cross-section has a variable area, one needs a certain resolution near the throat. A minimum of 30 points should be provided and some increase may be necessary if the time-dependent computation near the throat experiments difficulties.

Once the initial choice of mesh points has been made, no further intervention is necessary. Every time a new region is created, it starts with the minimum requirement of three points. As the width of the region increases, the number of mesh points is automatically increased to maintain an adequate resolution. Conversely, if the width of a region decreases, the number of mesh points is reduced until the minimum of three is eventually reached (an attempt to further reduce the number of mesh points results in detecting a reflection or an interaction, as explained in Section V).

IX. EXAMPLES AND DISCUSSION

The following examples intend to show the capabilities of the code in its present version and also to illustrate some of the points made in the preceding sections.

A. Cylindrical Ducts With Moving Pistons

Let us begin with the case of a cylindrical duct, bounded by two moving pistons, with the gas initially at rest.

In Run No. 2, whose results are shown in Fig. 6, the motion of the left piston is defined by:

$$x_b = 1 - \cos \pi t/2 \qquad t < 2$$
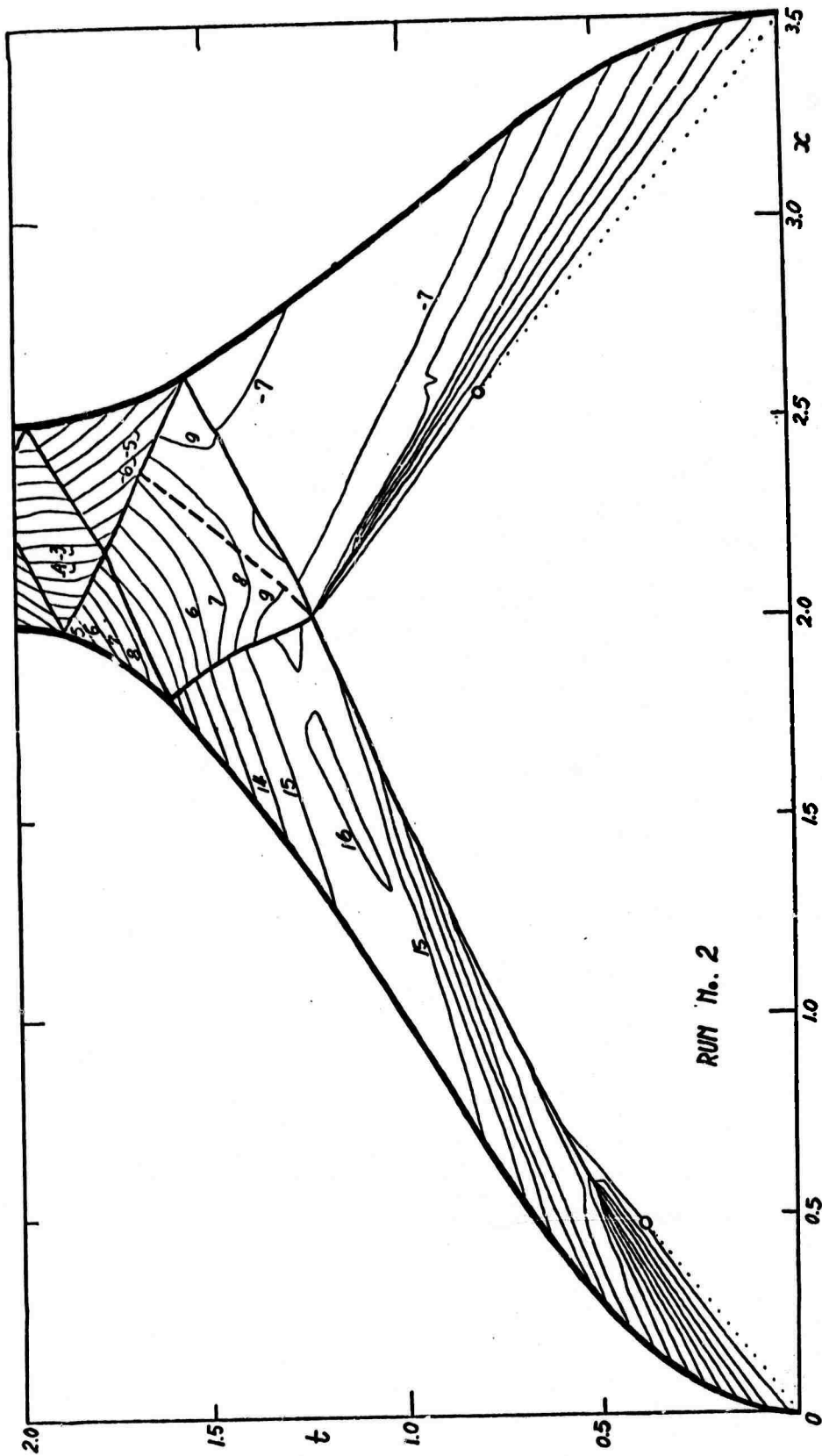$$x_b = 2 \qquad t > 2$$

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RUN No. 2

FIG. 6. FLOW PRODUCED BY TWO PISTONS MOVING AGAINST
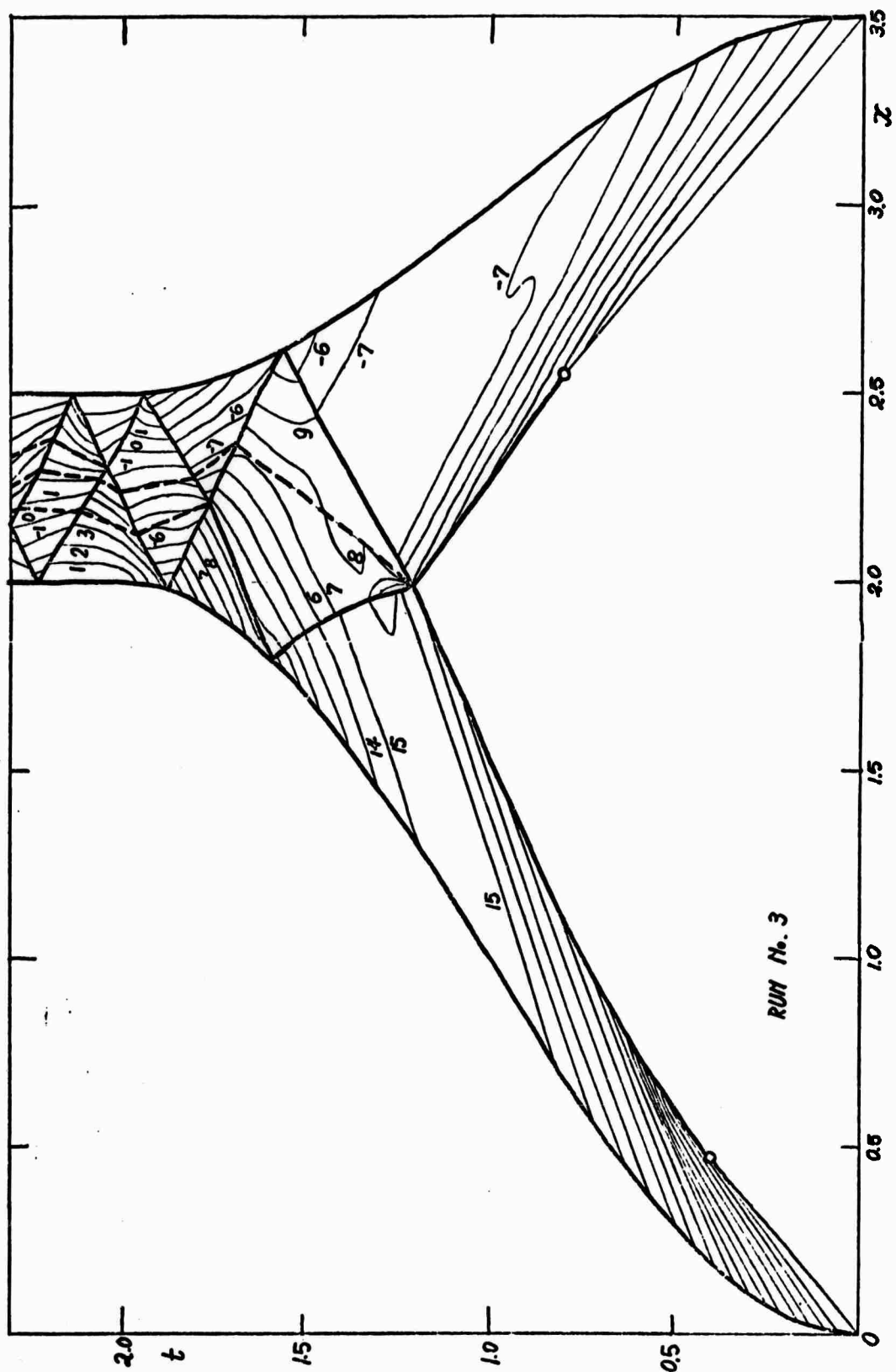EACH OTHER IN A CYLINDRICAL DUCT – FIRST COMPUTATION

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FIG. 7. FLOW PRODUCED BY TWO PISTONS MOVING AGAINST EACH
OTHER IN A CYLINDRICAL DUCT – IMPROVED COMPUTATION

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The motion of the right piston is defined by:

$$x_c = 3 + \tfrac{1}{2} \cos \pi t/2 \qquad\qquad t < 2$$

$$x_c = 2.5 \qquad\qquad t > 2$$

A single region is assumed initially, in which 49 points are computed
(including the boundary points). No provisions are made to define ficti-
tious shocks issuing from the pistons at t=0. Since the acceleration of
both pistons is different from zero at t=0, we must expect two gradient
discontinuities to propagate along characteristics issuing from x=0, t=0
and x=3.5, t=0. Both discontinuities evolve into shocks. According to
Eq. (39) of Ref. 7, the shock produced by the left piston should begin
at t=.4, x=.4732, the shock produced by the right piston should begin at
t=.8, x=2.5536. As pointed out in the preceding section, some inaccura-
cies are bound to appear before the shocks are formed; the shocks may not
be detected at the right places and the errors generated in the initial
phase of the computation will disturb the results in general. These
effects are evident in Fig. 6. The heavy solid lines are the piston and
shock trajectories; contact discontinuities are represented by heavy
broken lines; the lighter lines are lines of constant velocity. The two
circles indicate the theoretical origins of the two initial shocks. It
is clear that neither one of the shocks is detected properly and that
oscillations tend to appear. The shocks not being detected properly,
their path is slightly incorrect (the right one is displaced to the right
by about .05), their intersection does not occur at the right place and
the following pattern is incorrect. Nevertheless, the pattern is quali-
tatively significant, showing shock interactions and reflections at

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infinitum. Note the disappearance of the contact discontinuity as soon as the density jump across it becomes insignificant. The most important conclusion to be drawn from this computation is the waste of 49 mesh points to produce an imperfect result.

The same case is recomputed as Run No. 3, shown in Fig. 7. Now the gradient discontinuities are explicitly taken into account as dividers between regions, so that from the very beginning the computational field is split into three regions. The initial number of mesh points is now only 3. No oscillations appear, the coalescence of characteristics (represented by constant velocity lines in the original simple waves) occurs properly, and the shocks start gathering strength (and, consequently, bending their paths) at the theoretically predicted points. For this run, the tolerance on temperature ratio across a contact discontinuity was taken so small that no contact discontinuity is eliminated.

In Run No. 6, the motion of the left piston is defined by

$$x_b = 2t \qquad\qquad (t<.5)$$

$$x_b = -.25+t(3-t) \qquad\qquad (.5<t<1.5)$$

$$x_b = 2 \qquad\qquad (t>1.5)$$

and the motion of the right piston is defined by

$$x_c = 2.5 + \cos \pi t/2 \qquad\qquad (t<1)$$

$$x_c = 2.5+(\pi/2)(2.5-4t+1.5t^2) \qquad\qquad (1<t<4/3)$$

$$x = 2.23833 \qquad\qquad (t>4/3)$$

The gas is again initially at rest in a cylindrical duct. In this case, the left piston starts moving impulsively. A shock, binding two regions of uniform flow, originates at t=0, x=0 and moves at a constant speed
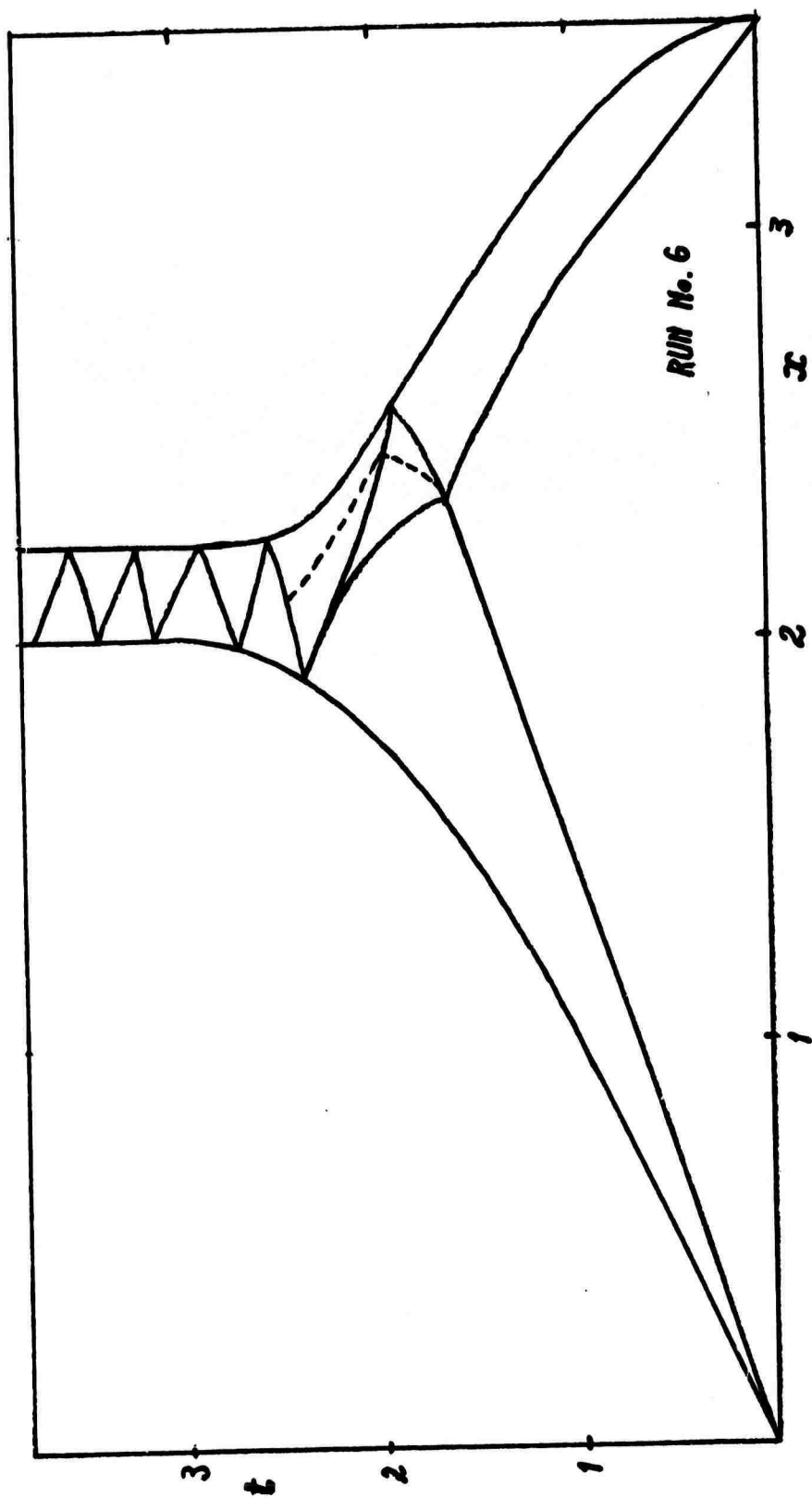
29

FIG. 8. FLOW PRODUCED BY TWO PISTONS MOVING AGAINST EACH OTHER

IN A CYLINDRICAL DUCT (LEFT PISTON STARTED IMPULSIVELY)

until it is overcome by the first expansion due to the braking of the left piston. The computed results are shown in Fig. 8, where only piston paths, shocks and contact discontinuities are plotted, for greater clarity.

B. Flow in Ducts of Variable Cross-section

The next experiment (Run No. 5) is made on a Laval nozzle, connecting an infinite capacity with an ambient where the pressure is .7 the stagnation pressure. The flow starts from a state of rest, as if a diaphragm were suddenly removed at the exit section. The geometry of the nozzle is defined by

$$A = x/2 + 1/x \qquad (0 < x < 3)$$

The results (isobars) are plotted in Fig. 9. It can be seen, from the figure, how a shock builds up in the divergent section of the nozzle and develops until it reaches a steady condition. The values at t=35 (not shown in the graph) are accurately representing a steady state.

Suppose now that the steady state, corresponding to the same Laval nozzle and the same pressure ratio, is the initial state and the pressure in the exit chamber is quickly raised to the stagnation value. Obviously, the final state should be a state of rest throughout the nozzle. The results of the computation for this case are shown in Fig. 10. In Fig.10, (Run No. 9, isobars plot) it can be seen how the pressure rise at the exit is propagated upstream through the subsonic portion of the flow, until it pushes the shock into the supersonic region. At about t=9.5, the shock has reached the vicinity of the mouth of the nozzle, and its strength has decreased so much that the shock is automatically eliminated by the program. An expanded region is left in the central portion of the
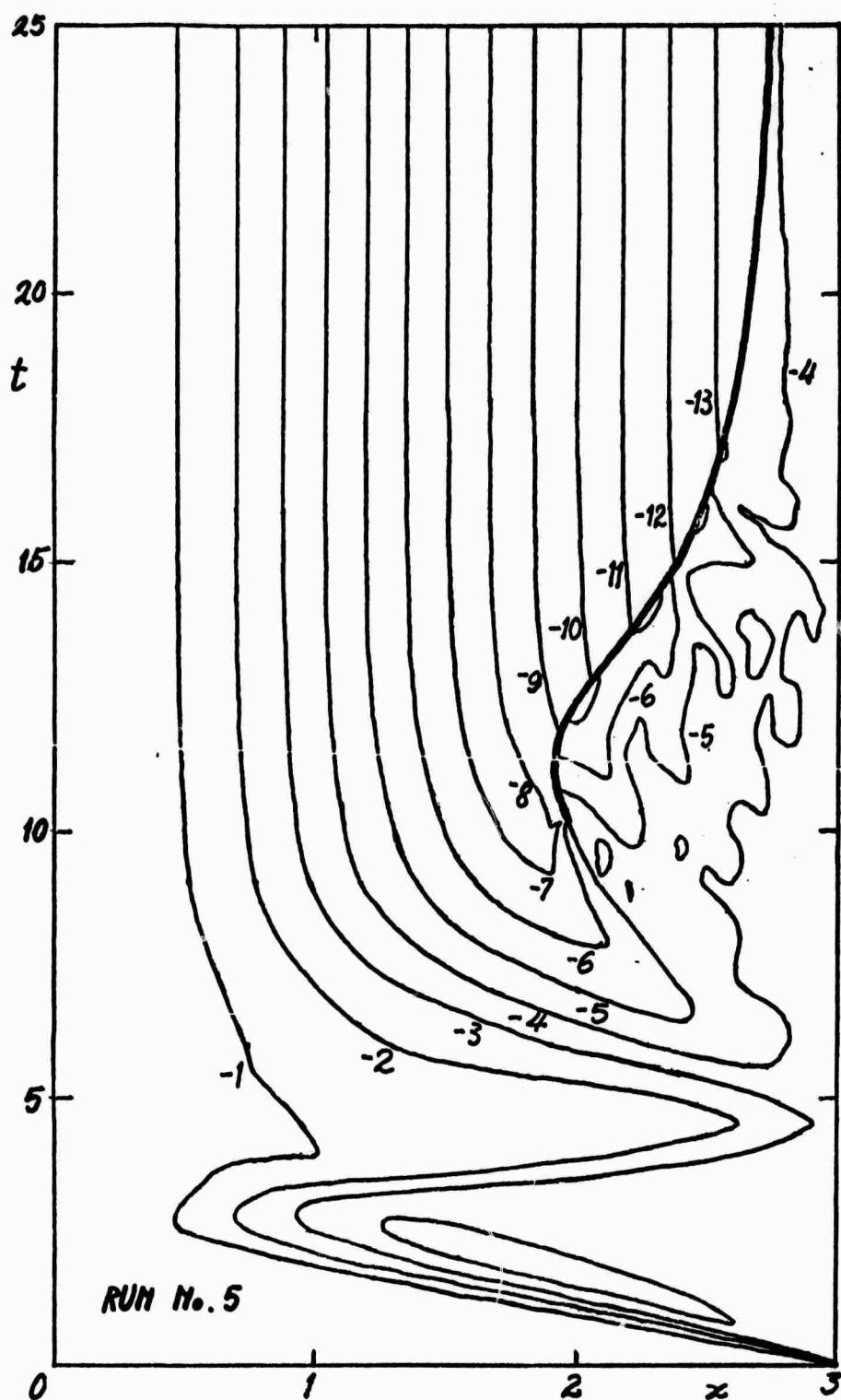
31

FIG. 9. FORMATION OF A SHOCK IN A LAVAL NOZZLE - MOTION STARTED
FROM REST BY REMOVING A DIAPHRAGM AT THE EXIT SECTION

nozzle, but the pressure slowly rises to its asymptotical stagnation value.

C. Propagation of Numerical Errors and Test for Steadiness

Fig. 11 is presented here to demonstrate how a study of isobar plots helps in finding programming mistakes. The figure refers to the same case described by Fig. 10. It is clear, though, that at the very beginning of the computation a disturbance originates at the entrance of the nozzle and propagates downstream. Note how a numerical disturbance propagates as a physical one. The disturbance seems to persist indefinitely; indeed, if it had existed at t=0 only, the isobars in the supersonic region should quickly come back to their original, steady state locations since they are not affected by the disturbance at the exit section until the shock reaches them. This is obviously not the case in the figure; all isobars settle down at a location displaced to the right of the initial one. The rise in pressure at the entrance mouth is also unrealistic. The error is rather small, though. Note that the initial perturbation passes through the shock but the shock itself is practically unaffected. All these facts pinpoint a mistake in the boundary conditions at the infinite capacity (x=0). Upon inspection of the program, it was found that the Mach number of the gas at rest had not been defined, and a value of .05, used for other purposes, had been forgotten where the exact value of 0 should have been stored. Fig. 10 is the result of the computation after correcting the mistake.

In connection with the problem of consistency of values computed according to the quasi-one-dimensional steady state with values obtained by applying the unsteady code to a set of steady initial and boundary
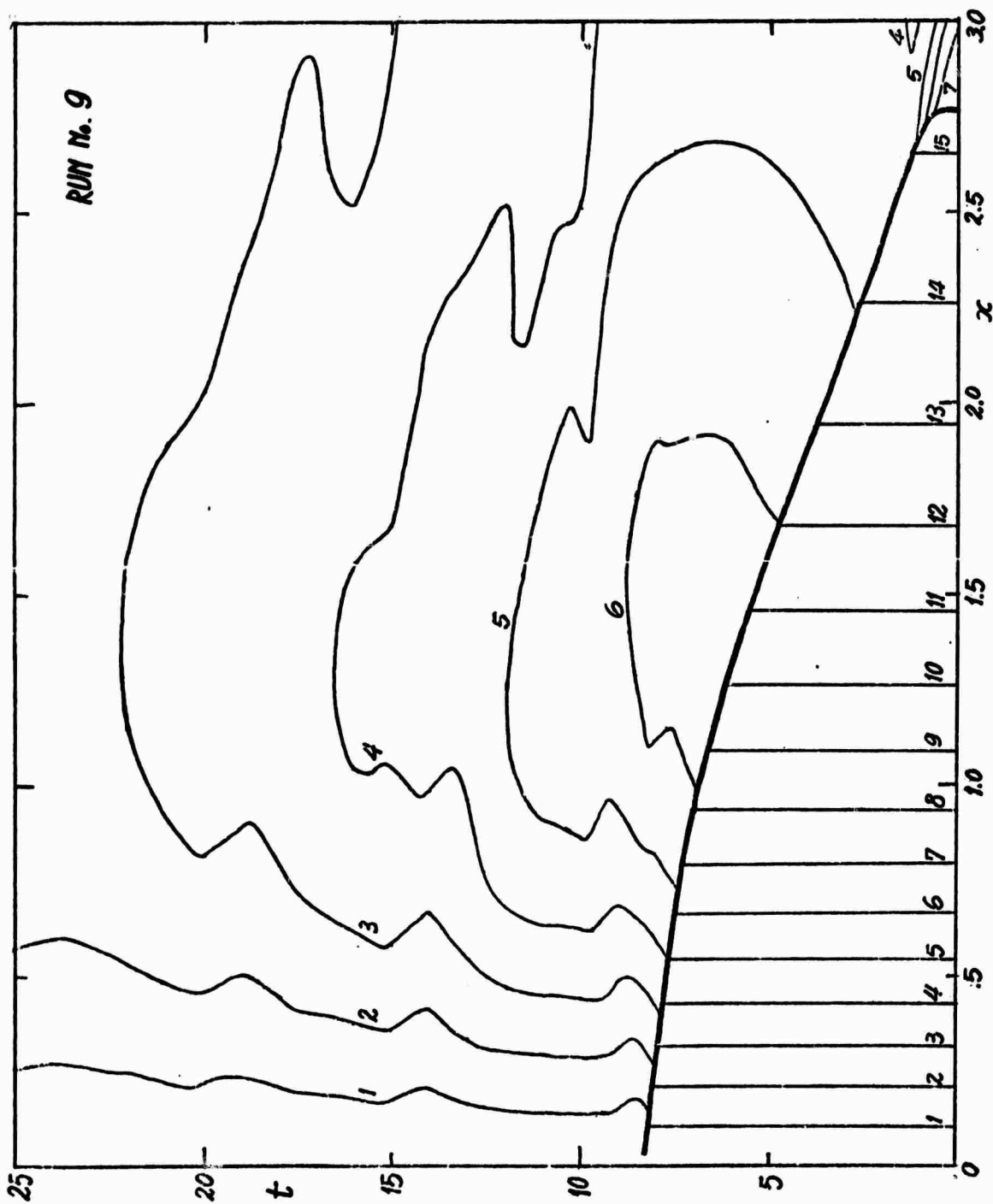
33

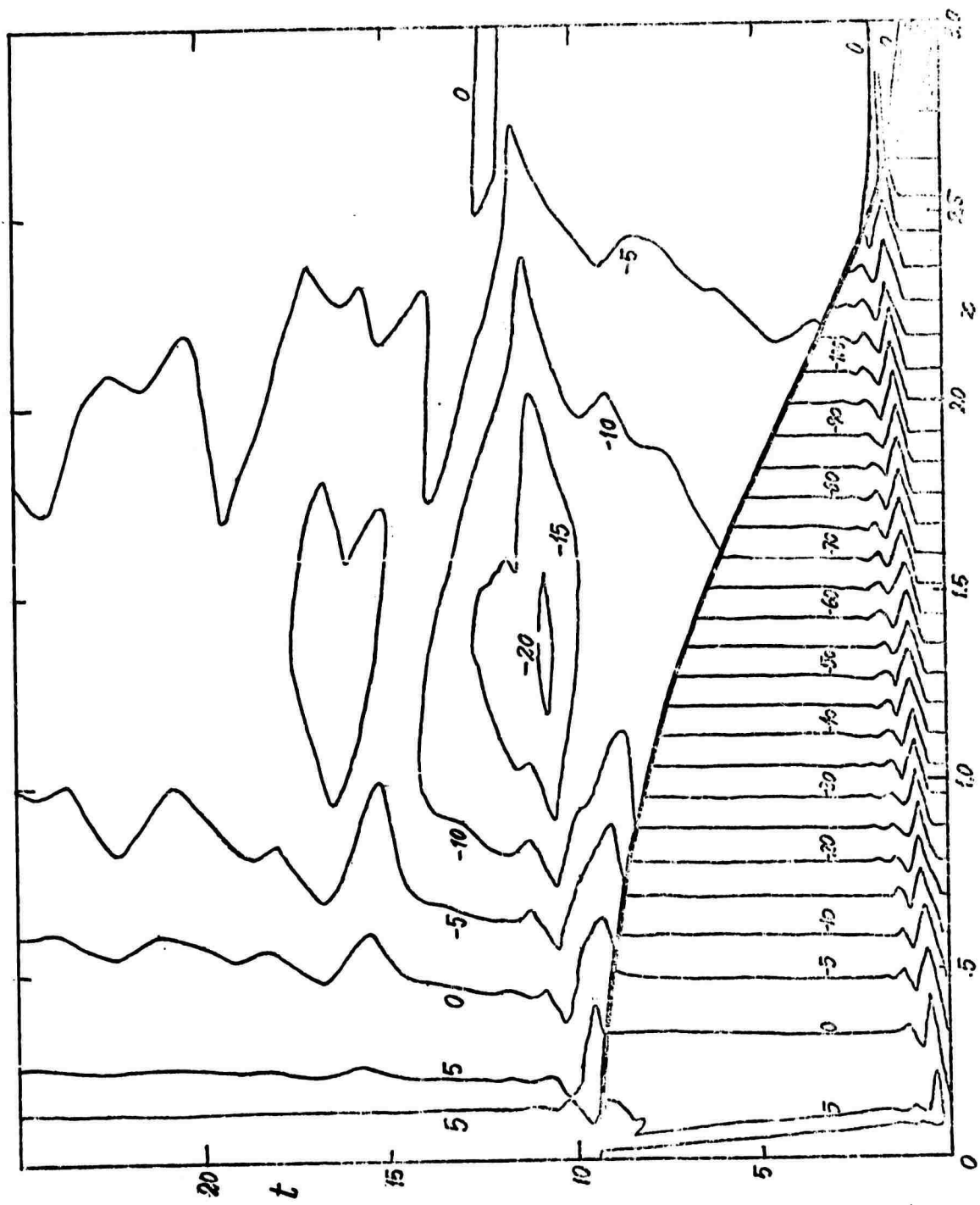FIG. 10. TRANSITION FROM STEADY FLOW TO REST IN A LAVAL NOZZLE

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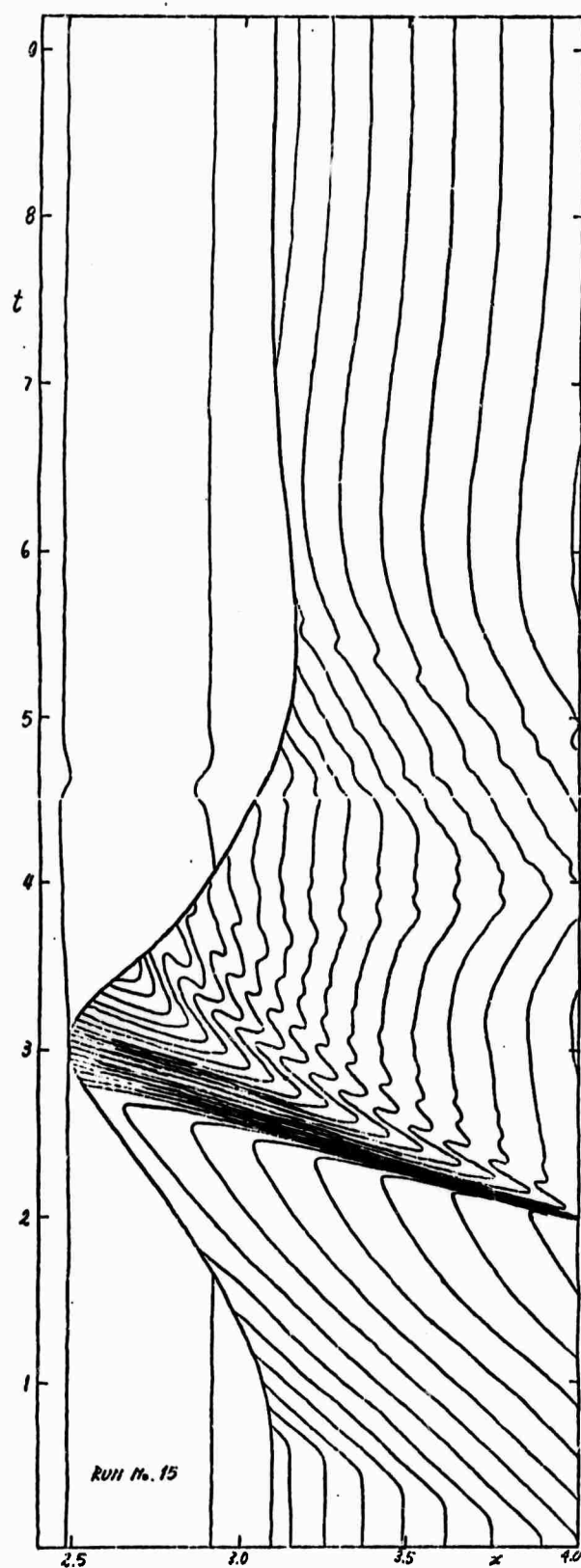FIG. 11. PROPAGATION OF AN ERROR IN THE COMPUTATION OF FIG. 10

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FIG. 12. RESPONSE OF A STABLE SHOCK TO A PRESSURE
PULSE AT THE EXIT OF A NOZZLE

36

conditions, a number of runs were made. One, for example, consists of
starting from the same initial conditions as in Run No. 9, but letting
the exit pressure unchanged. The pertinent isobar figure is not shown.
It consists of a set of straight lines, parallel to the t-axis, which
proves the perfect steadiness of the computation, despite the non-uniform
uniformity of values along the nozzle and the variations in cross-sectional
area (the run was executed for several hundred time steps with no changes
in the outputs).

A test of this kind should be conducted, prior to computing an
unsteady case for a variable area nozzle, in order to establish what is
the optimum number of mesh points to use.[*] If the mesh points are too
many, one wastes computational time; but if they are too few, the resolu-
tion may be affected. In the latter case, initial steady results would
not remain steady under steady boundary conditions. This is particularly
important in the vicinity of a throat, where changes in curvature of the
walls could make the time-dependent evaluation very delicate.

D. Engine Surge Simulation and Related Effects

An engine inlet is represented by a duct whose cross-sectional area
is defined by

$$A = 1 + \tfrac{1}{4}(x-2)^2 \qquad\qquad (0 \leq x \leq 4)$$

The Mach Number of the flow at the entry section is equal to 4, and the
ratio, p_{ex}^o, between exit pressure and entry pressure is 30. Two

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[*]All runs described under (B) have been made using 30 intervals along
the Laval nozzle.

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different steady states are possible, one with a shock in the divergent portion of the duct (at x=3.0948), which is a stable configuration, and another with a shock in the convergent portion of the duct (at x=.9052), which is an unstable configuration.

Starting from the stable configuration, the pressure at the exit section is let to vary according to the law

$$p_{ex} = p^O_{ex}(1 + \tfrac{1}{2} \sin t) \qquad (0 < t < 2)$$

(run No. 15).

At t=2, the exit pressure resumes its original value p^O_{ex}. The ensuing flow is shown in Fig. 12 (isobar plot) where only the portion of the duct affected by the perturbation is shown. The shock is initially pushed toward the throat, but then sucked·back by the rapid expansion. The interesting feature of this case is the smooth way in which the sudden jump in pressure at t=2 is negotiated by the computational technique, despite the presence of discontinuities in the first derivatives. It must be noted, however, that 20 nodes had to be used behind the shock to achieve this smooth result. With fewer nodes the transition produces an oscillating pattern which does not damp out. Another interesting feature is a slight oscillation, appearing at t=4.5 in the supersonic region. This is a purely numerical inaccuracy produced by an automatic change in the number of nodal points. Note how the numerical error is rapidly transmitted into and eliminated by the subsonic region.

Several other computations refer to the case in which the initial steady state describes the unstable configuration. First, the exit pressure variations defined by

$$p_{ex} = p^O_{ex} (1 + \tfrac{1}{2} \sin 2t) \qquad (0 < t < \pi/2)$$

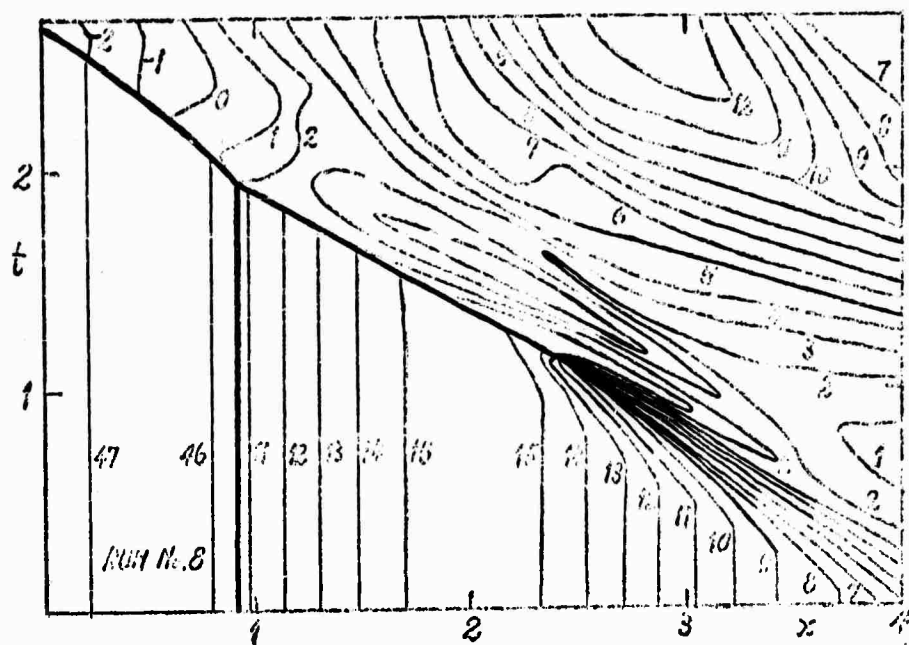is used (Run No. 8, Fig. 13, velocity plot). The initial pressure rise
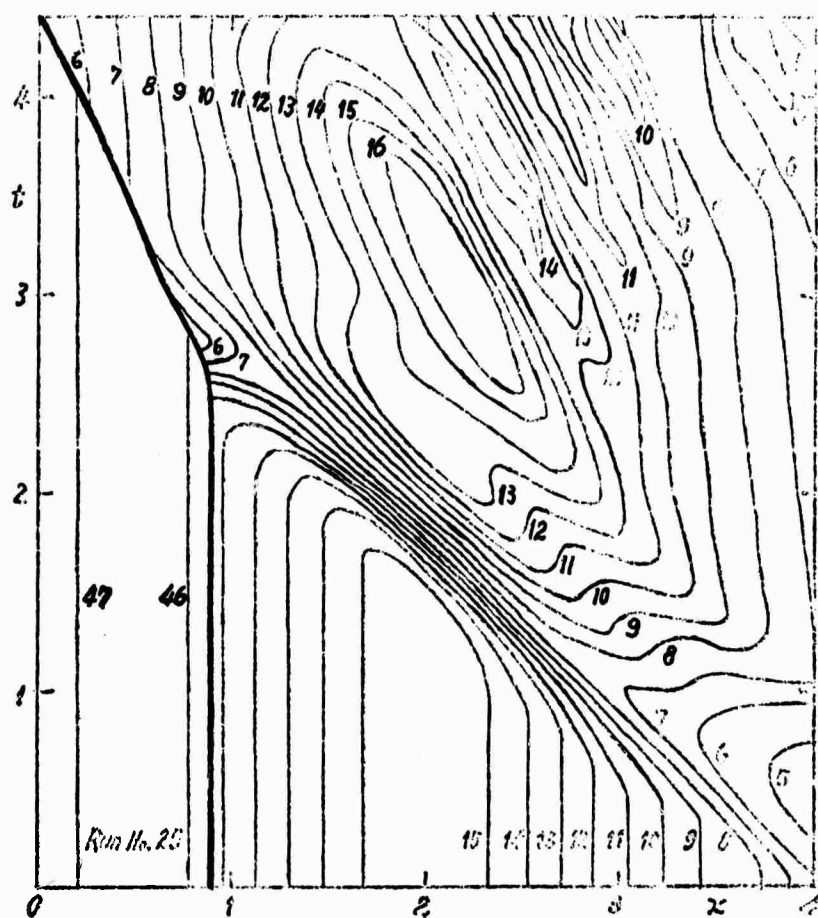
38

FIG. 13

FIG. 14

RESPONSE OF AN UNSTABLE SHOCK TO A PRESSURE
PULSE AT THE EXIT OF A NOZZLE

coalesces into a shock, which quickly overcomes the main shock, pushing it out of the duct. The following expansion does not catch up with the shock.

In a second problem (Run No. 25, Fig. 14, velocity plot) the exit pressure law is

$$p_{ex} = p_{ex}^{o} (1+.15 \sin 3t) \qquad (0 < t < \pi/3)$$

that is, a weaker perturbation, occurring in a shorter time. The compression waves do not coalesce into a shock but, again, the expansion is unable to recall the main shock into the duct.[*]

E. Instability of a Shock in the Convergent Section of a Supersonic Inlet

The last figures present the numerical description of the response of a shock to a perturbation in the impinging supersonic flow in the inlet defined in D.[**] The free stream pressure is defined by

$$P = \pm .15 \sin 3t \qquad (0 < t < \pi/3)$$
$$P = 0 \qquad (t > \pi/3)$$

J. 15 describes the flow corresponding to an increase in pressure. From the figure we see that the initial compression is propagated through the supersonic portion of the duct; the shock is pushed downstream, and the compression moves into the subsonic flow until it reaches the infinite capacity at the right end. There, it is reflected back as an expansion. This evolution, occurring in a time equal to about $\pi/6$, is followed by an expansion which, at the right end of the duct, reflects back as a compression. The latter acquires strength during its upstream motion until

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[*] Both runs No. 8 and No. 25 have been made with a total of 30 points over the entire region.

[**] These runs have been made with 30 mesh points over the entire duct.
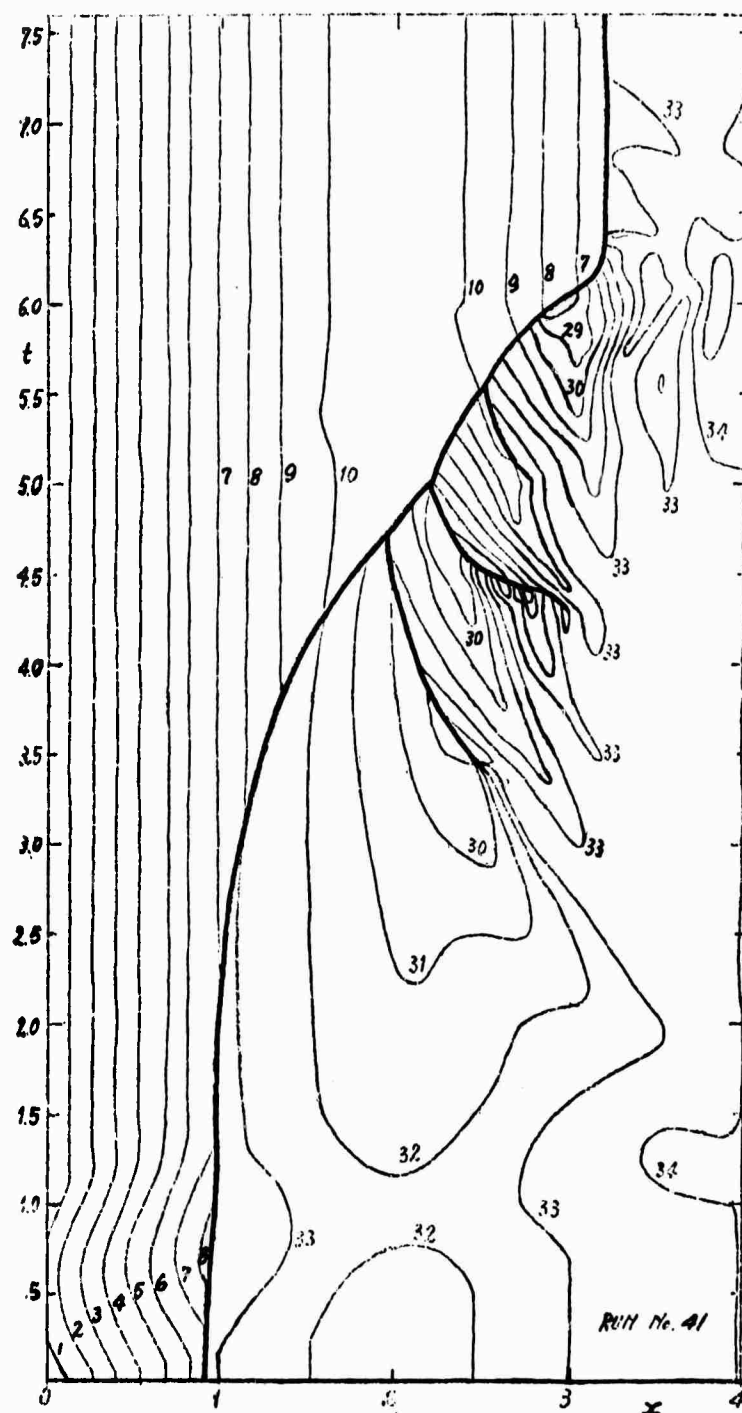
40

FIG. 15. RESPONSE OF AN UNSTABLE SHOCK TO A COMPRESSION
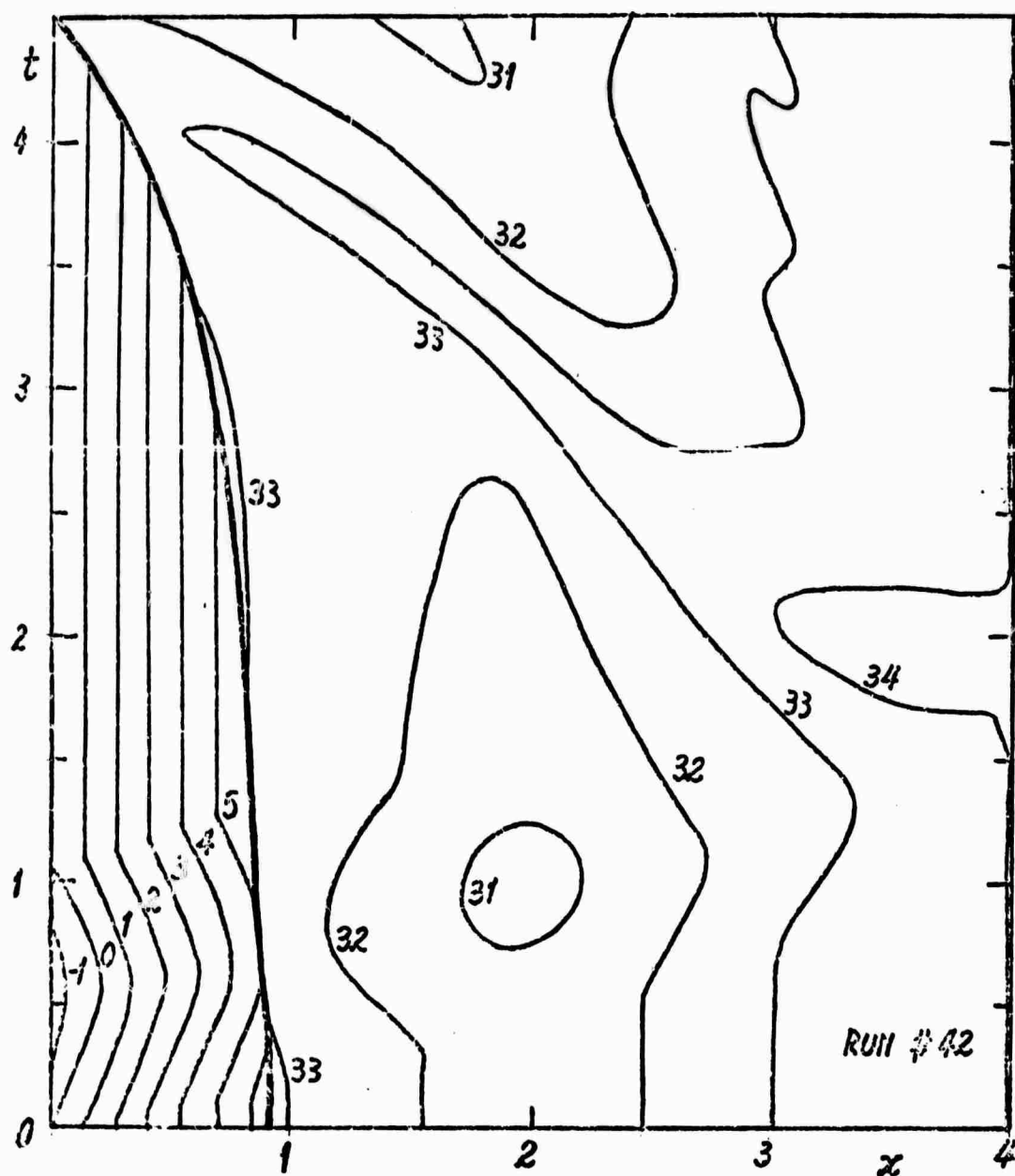FOLLOWED BY AN EXPANSION IN THE MAIN STREAM

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FIG. 16. RESPONSE OF AN UNSTABLE SHOCK TO AN EXPANSION

FOLLOWED BY A COMPRESSION IN THE MAIN STREAM

it coalesces into a shock. However, the main shock cannot be switched back to its original position. Indeed, the upstream supersonic flow is quickly reestablished into a practically steady state. The shock, thus, occurs at a lower Mach number (made even lower by the motion of the shock in the direction of the flow) and keeps losing strength until it reaches the throat. From then on, the shock gathers strength again and eventually it locks in the stable configuration mentioned in D. Between t=3.5 and t=6.5, we note a certain deterioration in the computation behind the main shock. This is only due to lack of resolution; the number of mesh points is indeed too small (about a dozen) behind the secondary shock. However, strong compressions are absorbed by several extra shocks, and the pattern is quickly restored to normality. The figure gives, thus, a good example of the usage of shocks to represent strong compressions without creating instabilities (a purely numerical device which is much safer and economical, and less dissipative, than artificial viscosity).

Fig. 16 (Run No. 42, isobars plot) describes the opposite case of a decrease in pressure in the free stream. The initial expansion makes the shock move upstream. When the supersonic flow recovers its steadiness, the shock occurs at a higher Mach number, that is, with greater strength, and it cannot be recalled back.

The last two figures, Figs. 17 and 18 (Run Nos. 43 and 44, isobars plots), confirm the stability of the shock in the divergent section of the duct; the first under a compressi   followed by an expansion, the second in the opposite case. Compare the two figures to note how faster the compression propagates with respect to the expansion. Compare Fig. 15 and Fig. 17, Fig. 16 and Fig. 18 to note how the initial pattern, which
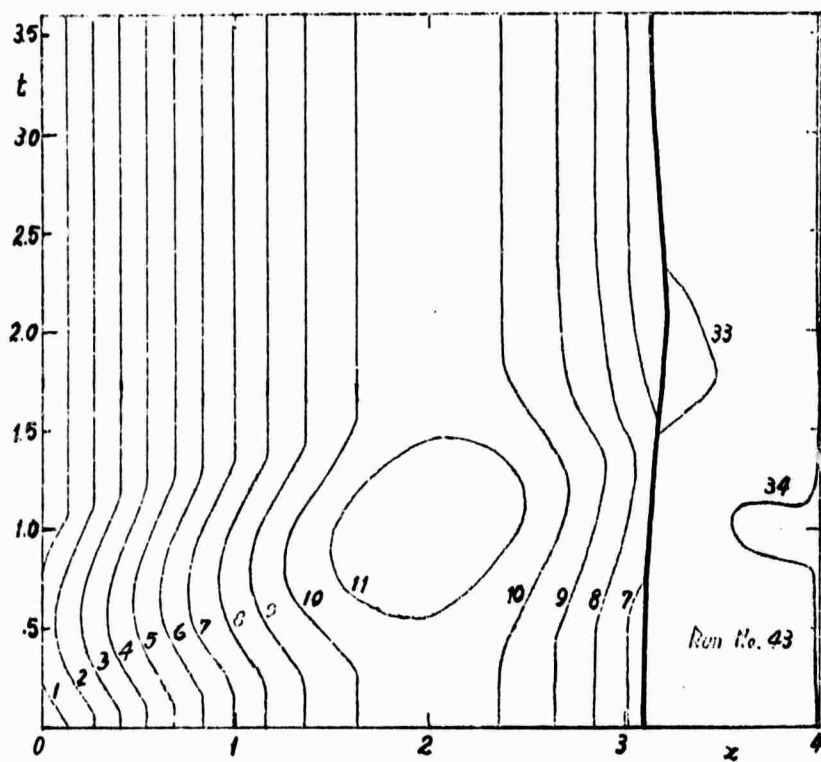
43

FIG. 17. RESPONSE OF A STABLE SHOCK TO A COMPRESSION
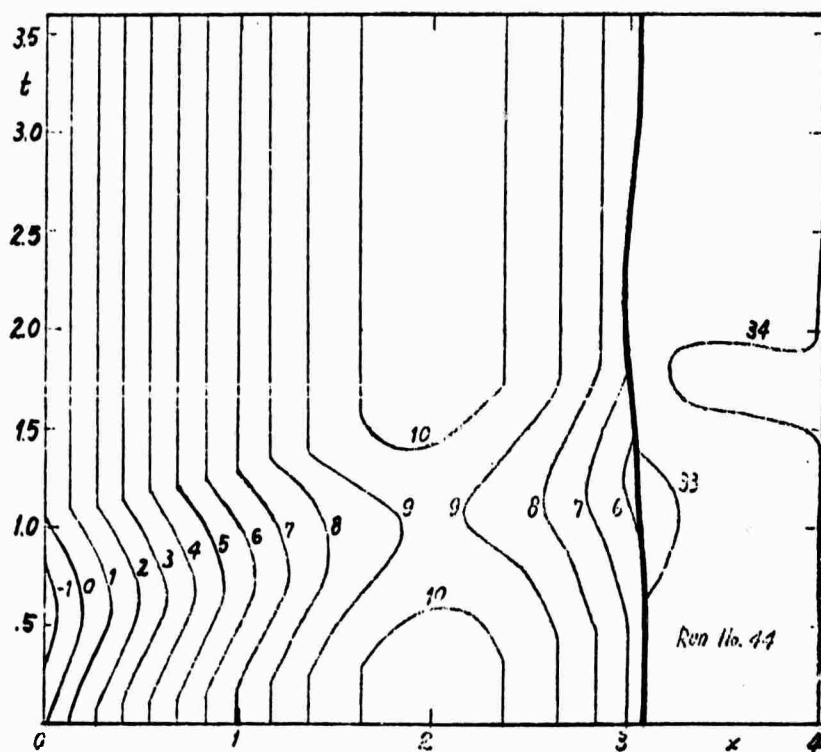FOLLOWED BY AN EXPANSION IN THE MAIN STREAM

FIG. 18. RESPONSE OF A STABLE SHOCK TO AN EXPANSION
FOLLOWED BY A COMPRESSION IN THE MAIN STREAM

describes a compression (expansion) in the throat region in a subsonic flow, is the opposite of the pattern which describes a similar evolution in a supersonic flow.

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    Toronto Press, Canada, 1970.

NOTE:

The present paper is an attempt to illustrate the philosophy and
the basic ideas of a computational program. No important detail has
been omitted. However, a minute description of all intricacies and sub-
tleties of the code would make the paper three or four times as longer.
The author believes that too many details would confuse the issues for
a reader interested in the nature of the approach, and would be required
only by a restricted number of readers. The author will be glad to pro-
vide additional explanations to anyone who so desires.